ON THE SOLUTION OF NONSTATIONARY HEAT-CONDUCTION PROBLEMS WITH VARIABLE HEAT-TRANSFER COEFFICIENT

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The article describes an exact method for calculating the temperature field in solids when they are heated in a medium with a variable heat-transfer coefficient and a nonuniform initial temperature distribution.

In [1] a method for the exact calculation of the temperature field of a solid object undergoing heat exchange in a medium with a variable temperature and a variable heat-transfer coefficient was discussed for a large number of Bi(Fo) functions of practical interest, as applied to an infinite plate. For $\theta(1, \mathrm{Fo})$, the temperature of the heated surface, we found in [1] an ordinary differential equation with variable coefficients which is solvable by operational methods [2]. The initial temperature distribution was assumed to be zero. We shall now show, using the example of a plate, how to deal with a nonuniform initial distribution. We shall assume that the temperature of the medium is zero. Heat transfer takes place at the plate surface X = 1, while the surface X = 0 is thermally insulated.

To solve the problem, we must establish how $\partial\Theta(1, \mathrm{Fo})/\partial X$ varies with $\Theta(1, \mathrm{Fo})$.

It was shown in [3] that if Fo > 0, the function $\partial\Theta(1, \mathrm{Fo})/\partial X$ can be represented as a convergent series

$$\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X}=\sum_{i=1}^{\infty} Z_i(\mathrm{Fo}), \tag{1}$$

in which $Z_i(\mathrm{Fo})$, i = 1, 2, . . . , are determined from the solution of the ordinary differential equations

$$T_i\dot{Z}_i(\mathrm{Fo})+Z_i(\mathrm{Fo})=2T_i\dot{\Theta}(1,\ \mathrm{Fo}),\ i=1,\ 2,\ ..,\tag{2}$$

with initial conditions $Z_i(0) = Z_i^0$, uniquely determined by the initial temperature distribution function. For the equations in (2) we have

$$T_i=\frac{4}{(2i-1)^2\pi^2}.$$

The solutions $Z_i(\mathrm{Fo})$ of these equations with initial conditions Z_i^0 which are nonzero at time Fo = 0 − 0 (before the start of the perturbation) will be identical for Fo ≥ 0 + 0 (after the start of the perturbation) with the solutions $y_i(\mathrm{Fo})$ of the equations

$$T_i\dot{y}_i(\mathrm{Fo})+y_i(\mathrm{Fo})=2T_i\dot{\Theta}(1,\ \mathrm{Fo})+T_iZ_i^0\,\delta(\mathrm{Fo}),\ i=1,\ 2,\ ..,\tag{3}$$

with initial conditions which are zero at time Fo = 0 − 0 [4]. Here $\delta(\mathrm{Fo})$ is the Dirac δ-function.

Summation of the left and right sides of Eq. (3), taking account of (1) and the identities $Z_i(\mathrm{Fo}) \equiv y_i(\mathrm{Fo})$, which are valid for Fo ≥ 0 + 0, yields:

$$-\sum_{i=1}^{\infty}T_i\dot{y}_i(\mathrm{Fo})+\dot{\Theta}(1,\ \mathrm{Fo})\sum_{i=1}^{\infty}2T_i=\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X}-\delta(\mathrm{Fo})\sum_{i=1}^{\infty}T_iZ_i^0.\tag{4}$$

Now we multiply each equation of (3) by T_i and differentiate term by term:

$$T_i^2\ddot{y}_i(\mathrm{Fo})+T_i\dot{y}_i(\mathrm{Fo})=2T_i^2\ddot{\Theta}(1,\ \mathrm{Fo})+T_i^2Z_i^0\,\dot{\delta}(\mathrm{Fo}),\ i=1,\ 2,\ \ldots\tag{5}$$

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Summing with respect to i in (5), we obtain the sum $\sum_{i=1}^{\infty} T_i \dot{y}_i(\mathrm{Fo})$ and substitute the resulting expression into (4):

$$\sum_{i=1}^{\infty} T_i^2 \ddot{y}_i(\mathrm{Fo}) - \ddot{\Theta}(1,\ \mathrm{Fo}) \sum_{i=1}^{\infty} 2T_i^2 + \dot{\Theta}(1,\ \mathrm{Fo}) \sum_{i=1}^{\infty} 2T_i = \frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} - \delta(\mathrm{Fo}) \sum_{i=1}^{\infty} T_i Z_i^0 + \dot{\delta}(\mathrm{Fo}) \sum_{i=1}^{\infty} T_i^2 Z_i^0. \quad (6)$$

Proceeding with repeated transformations of this kind we finally obtain an ordinary differential equation for $\Theta(1,\ \mathrm{Fo})$:

$$\sum_{n=1}^{\infty} a_n \frac{d^n}{d\mathrm{Fo}^n} \Theta(1,\ \mathrm{Fo}) = \frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} + \sum_{m=0}^{\infty} b_m \frac{d^m}{d\mathrm{Fo}^m} \delta(\mathrm{Fo}) \quad (7)$$

with the coefficients

$$a_n = (-1)^{n+1} \sum_{i=1}^{\infty} 2T_i^n,\ n = 1,\ 2,\ \ldots \quad (8)$$

$$b_m = (-1)^{m+1} \sum_{i=1}^{\infty} Z_i^0\, T_i^{m+1},\ m = 0,\ 1,\ \ldots \quad (9)$$

Taking account of the boundary condition of the third kind for the case of a medium at zero temperature,

$$-\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial X} = \mathrm{Bi}(\mathrm{Fo})\,\Theta(1,\ \mathrm{Fo}), \quad (10)$$

we finally arrive at the equation

$$\mathrm{Bi}(\mathrm{Fo})\,\Theta(1,\ \mathrm{Fo}) + \sum_{n=1}^{\infty} a_n \frac{d^n}{d\mathrm{Fo}^n} \Theta(1,\ \mathrm{Fo}) = \sum_{m=0}^{\infty} b_m \frac{d^m}{d\mathrm{Fo}^m} \delta(\mathrm{Fo}). \quad (11)$$

It follows from the method used for obtaining Eq. (11) that here the initial conditions for $\mathrm{Fo} = 0-0$ will be zero.

Assume, as in [1], that

$$\mathrm{Bi}(\mathrm{Fo}) = \mathrm{Bi}_0 - f_0(\mathrm{Fo}), \quad (12)$$

where $\mathrm{Bi}_0 = \mathrm{const}$ and $f_0(\mathrm{Fo})$ is representable by a rational combination of sines (or cosines), polynomials, and exponents.

Proceeding in a manner analogous to [1], for a solution of Eq. (11), in the image domain, we make use of the "bifrequency transfer function" method of [2]. According to [2],

$$\overline{\Theta}(1,\ s) = \underset{\mathrm{Fo}\to s}{L}\ \Theta(1,\ \mathrm{Fo}) = \sum_{q=q_j} \frac{1}{(\gamma_j - 1)!} \frac{d^{\nu_j - 1}}{dq^{\nu_j - 1}} [(q - q_j)^{\nu_j} W(s,\ q)], \quad (13)$$

where the sum is taken over all the q_j-poles of the second argument of the function $W(s,\ q)$, and ν_j is the multiplicity of these poles.

If (12) is satisfied, we can obtain the bifrequency transfer function $W(s,\ p)$ in the form of the absolutely and uniformly convergent series

$$W(s,\ p) = \sum_{\nu=0}^{\infty} W_\nu(s,\ p). \quad (14)$$

In the problem under consideration the zeroth term of this series yields the formula

$$W_0(s,\ p) = \frac{1}{p\Psi(s)} \sum_{k=0}^{\infty} b_k s^k, \quad (15)$$

where

$$\Psi(s) = \mathrm{Bi}_0 + \sum_{k=1}^{\infty} a_k s^k, \tag{16}$$

and the a_k and b_k are the coefficients (8) and (9) of Eq. (11).

It should be noted that the sum in (16) is a series expansion of the function $\sqrt{s}\,\mathrm{th}\,\sqrt{s}$, and in (15)

$$\sum_{k=0}^{\infty} b_k s^k = -\sum_{i=0}^{\infty} \frac{Z_i^0}{s + \frac{1}{T_i}}. \tag{17}$$

Therefore

$$W_0(s, p) = -\frac{1}{p\Psi(s)} \sum_{i=0}^{\infty} \frac{Z_i^0}{s + \frac{1}{T_i}}, \tag{18}$$

$$\Psi(s) = \mathrm{Bi}_0 + \sqrt{s}\,\mathrm{th}\sqrt{s}. \tag{19}$$

All the subsequent ($\nu = 1, 2, \ldots$) terms of the series (14) are found by the recursion formula:

$$W_\nu(s, p) = \sum_{q=q_j} \frac{1}{(\nu_j - 1)!} \frac{d^{\nu_j - 1}}{dq^{\nu_j - 1}} [(q - q_j)^{\nu_j} W_u(s, q) W_{\nu-1}(s - q, p - q)]. \tag{20}$$

The sum in (20) is taken over all the q_j-poles of multiplicity ν_j of the second argument of the bifrequency transfer function $W_u(s, q)$, which in our problem has the form

$$W_u(s, q) = \frac{F_0(q)}{\Psi(s)}, \quad F_0(q) = \underset{F_0 \to q}{L} f_0(\mathrm{Fo}). \tag{21}$$

After determining the temperature $\Theta(1, \mathrm{Fo})$, the temperature field of the plate $\Theta(X, \mathrm{Fo})$ can be found from the solution of the problem with a boundary condition of the first kind.

NOTATION

Θ	is the temperature;
L	is the thickness of plate;
x	is the space coordinate;
a	is the thermal diffusivity;
λ	is the thermal conductivity;
α	is the heat-transfer coefficient;
t	is the time;
$X = x/L$	is the dimensionless coordinate;
$\mathrm{Fo} = at/L^2$	is the Fourier number;
$\mathrm{Bi}(\mathrm{Fo}) = \alpha(\mathrm{Fo})L/\lambda$	is the Biot number.

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